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High Performance Computing for Engineering Problems

Classification Of Partial Differential Equations And Their Solution Characteristics

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Contents

Partial Differential Equations

- An equation which involves several independent variables (usually denoted *x, y, z, t, …..*), a dependent function *u* of these variables, and the partial derivatives of the dependent function *u* with respect to the independent variables such as

  $F(x, y, z, t, \ldots\ldots, u_x, u_y, u_z, u_t, \ldots\ldots, u_{xx}, u_{yy}, \ldots\ldots, u_{xy}, \ldots\ldots)=0$

  is called a partial differential equation.
- Partial differential equations are used to formulate, and thus aid the solution of, problems involving functions of several variables; such as the propagation of sound or heat, electrostatics, electrodynamics, fluid flow, and elasticity.

Partial Differential Equations cont.

- Examples:

i. $u_t = k(u_{xx} + u_{yy} + u_{zz})$ [linear three-dimensional heat equation]

ii. $u_{xx} + u_{yy} + u_{zz} = 0$ [Laplace equation in three dimensions]

iii. $u_{tt} = c^2(u_{xx} + u_{yy} + u_{zz})$ [linear three-dimensional wave equation]

iv. $u_t + uu_x = \mu u_{xx}$ [nonlinear one-dimensional Burger equation]

Partial Differential Equations cont.

- The order of a partial differential equation is the order of the highest derivative occurring in the equation.
- All the above examples are second order partial differential equations.
- $u_t = u u_{xxx} + \sin x$ is an example for third order partial differential equation.

Ordinary Differential Equations vs. Partial Differential Equations

Partial Differential Equations

- A relatively simple partial differential equation is

$$u_x(x, y)=0$$

- General solution of the above equation is

$$u(x, y)=f(y)$$

- General solution involves arbitrary functions

Ordinary Differential Equations

- The analogous ordinary differential equation is

$$u'(x)=0$$

- General solution of the above equation is

$$u(x)=c$$

- General solution involves arbitrary constants

Linear Partial Differential Equations

- The equation is called linear if the unknown function only appears in a linear form.

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = d(x, y)$$

- Almost linear partial differential equations

$$P(x, y)u_x + Q(x, y)u_y = R(x, y, u)$$

- Quasi-linear partial differential equations

$$P(x, y, u)u_x + Q(x, y, u)u_y = R(x, y, u)$$

Classification of second order linear PDEs

Consider the second order linear PDE in two variables

$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$ (1)

The discriminant

$$d = B^2(x_0, y_0) - 4A(x_0, y_0)C(x_0, y_0)$$

At (x_0, y_0), the equation is said to be

- Elliptic if $d<0$
- Parabolic if $d=0$
- Hyperbolic if $d>0$

If this is true at all points in a domain Ω, then the equation is said to be elliptic, parabolic, or hyperbolic in that domain

Classification of second order linear PDEs cont.

- If there are n independent variables x_1, x_2 , ..., x_n, a general linear partial differential equation of second order has the form
- $\sum\sum a_{i,j} u_{x_i x_j}$ plus lower order terms =0
- The classification depends upon the signature of the eigenvalues of the coefficient matrix.

Classification of second order linear PDEs cont.

i. Elliptic: The eigenvalues are all positive or all negative.
ii. Parabolic : The eigenvalues are all positive or all negative, save one which is zero.
iii. Hyperbolic: There is only one negative eigenvalue and all the rest are positive, or there is only one positive eigenvalue and all the rest are negative.

Canonical Forms

- Transformation of independent variables x and y of eq.(1) to new variables ξ, η, where

$$\xi = \xi(x, y),\ \eta = \eta(x, y)$$

i. Elliptic: $u_{\xi\xi} + u_{\eta\eta} = \varphi(\xi, \eta, u, u_{\xi}, u_{\eta})$

ii. Parabolic: $u_{\xi\xi} = \varphi(\xi, \eta, u, u_{\xi}, u_{\eta})$ or $u_{\eta\eta} = \varphi(\xi, \eta, u, u_{\xi}, u_{\eta})$

iii. Hyperbolic: $u_{\xi\xi} - u_{\eta\eta} = \varphi(\xi, \eta, u, u_{\xi}, u_{\eta})$ or $u_{\xi\eta} = \varphi(\xi, \eta, u, u_{\xi}, u_{\eta})$

Characteristics

- Consider $L[u]=f(x, y, u, u_x, u_y)$ --(2) where $L[u]=a(x, y)u_{xx} + b(x, y)u_{xy} + c(x, y)u_{yy}$
- $L[u]$ is the principle part of the equation
- $\xi=\xi(x, y), \eta=\eta(x, y)$
- Transformed equation: $M[u]=g(\xi, \eta, u, u_\xi, u_\eta)$ with principle part

$M[u]=A(\xi, \eta)u_{\xi\xi} + B(\xi, \eta)u_{\xi\eta} + C(\xi, \eta)u_{\eta\eta}$ where

$$A=a\xi_x^2 + b\xi_x\xi_y + c\xi_y^2$$

$$B=2a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + 2c\xi_y\eta_y$$

$$C=a\eta_x^2 + b\eta_x\eta_y + c\eta_y^2$$

Characteristics cont.

- An integral of an ordinary differential equation is a function φ whose level curves, $\varphi(x, y)=k$, characterize solutions of the equation implicitly.
- $a(x, y)\xi_x^2 + b(x, y)\xi_x\xi_y + c(x, y)\xi_y^2=0$ iff ξ is an integral of the ordinary differential equation

$$a(x, y)y'^2 - b(x, y)y' + c(x, y)=0 \quad \text{--(3)}$$

$$\Rightarrow y'=[b(x, y) \pm \{b^2(x, y) - 4a(x, y)c(x, y)\}^{1/2}]/2a$$

- An integral curve, $\varphi(x, y)=k$, of (3) is a characteristic curve, and (3) is called the characteristic equation for the partial differential equation (2)

Characteristics cont.

- Therefore,

i. Elliptic partial differential equations have no characteristic curves

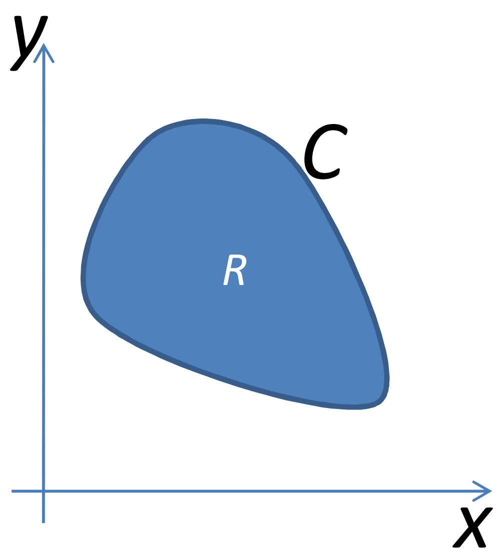
ii. Parabolic partial differential equations have a single characteristic curve

iii. Hyperbolic partial differential equations have two characteristic curves

Initial and Boundary Conditions

(a) Elliptic Equations: Boundary conditions

e.g. $u_{xx} + u_{yy} = G$ in a finite region R bounded by a closed curve C.



Initial and Boundary Conditions cont.

We must specify

(i) u on curve C or

(ii) u_n on C (**n** is outward normal to C) or

(iii) $\alpha u + \beta u_n$ on C (α and β are given constants) or

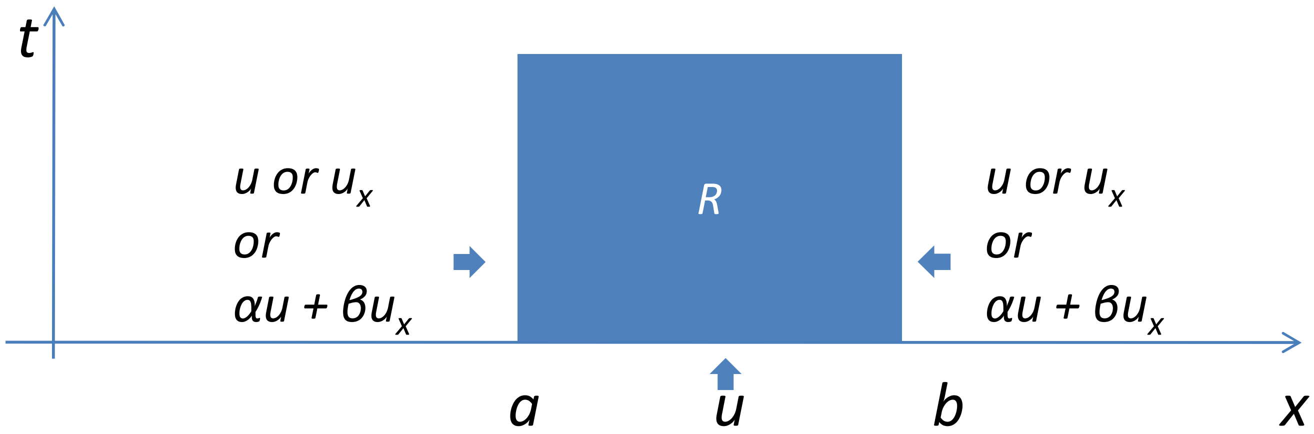
(iv) a combination of (i), (ii) and (iii) on different parts of C

- In Cartesian coordinates the simplest case is if R is rectangular with boundary condition (i).
- R can extend to infinity, in which case we must specify how the solution behaves as x or y (or both x and y) tend to infinity.

Initial and Boundary Conditions cont.

(b) Parabolic Equations: Initial conditions and boundary conditions.

e.g. $u_{xx}=u_t$ in the open region R in the (x, t) plane. R is the region $a \leq x \leq b, 0 \leq t < \infty$



Initial and Boundary Conditions cont.

- We must specify u on $t=0$ (i.e. $u(x, 0)$) for $a \le x \le b$. This is an initial condition (e.g. an initial temperature distribution) and suitable boundary conditions x on a and b are as shown.

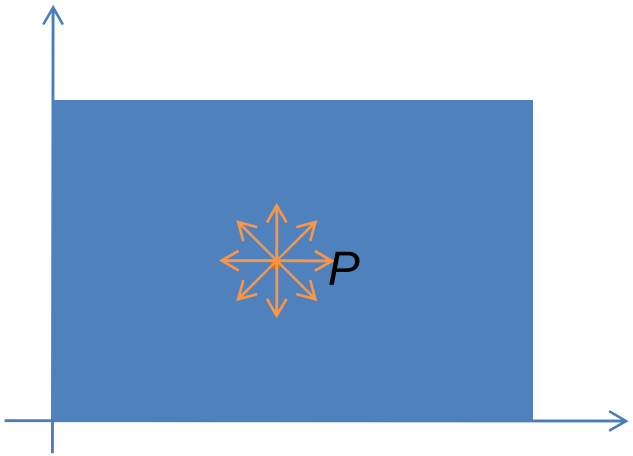
(c) Hyperbolic Equations: e.g. $u_{xx}=u_{tt}$ Initial conditions and boundary conditions as for (b) except that we must also specify u_t at $t=0$ for $a \le x \le b$ (in addition to u) and R is the region $a \le x \le b$, $-\infty < t < \infty$

Elliptic Partial Differential Equations

- The discriminant $B^2 - 4AC < 0$
- Solutions of elliptic PDEs are as smooth as the coefficients allow, within the interior of the region where the equation and solutions are defined.
- For example, solutions of Laplace's equation are analytic within the domain where they are defined, but solutions may assume boundary values that are not smooth.

Elliptic Partial Differential Equations cont.

- Region of Influence: Entire domain
- Region of Dependence: Entire domain



- Any disturbance at *P* is felt throughout the domain

Elliptic Partial Differential Equations cont.

Examples:

(i) Laplace Equation: $\Delta u=0$

- The Laplace equation is often encountered in heat and mass transfer theory, fluid mechanics, elasticity, electrostatics, and other areas of mechanics and physics.
- The two dimensional Laplace equation has the following form:

$u_{xx} + u_{yy}=0$ in the Cartesian coordinate system,

$(1/r)(ru_r)_r +(1/r^2)u_{\vartheta\vartheta}=0$ in the polar coordinate system

Laplace Equation cont.

- A function which satisfies Laplace's equation is said to be harmonic.
- A solution to Laplace's equation has the property that the average value over a spherical surface is equal to the value at the center of the sphere (Gauss’ harmonic function theorem).
- Solutions have no local maxima or minima.
- Because Laplace's equation is linear and homogeneous, the superposition of any two solutions is also a solution

Laplace Equation cont.

Solution of Laplace's equation:

Consider $u_{xx} + u_{yy}=0$ (2)

Solve by separation of variables

Let $u=X(x)Y(y)$

Substituting it in (2), we get

$$(1/X)X''=-(1/Y)Y''=k$$

Solution of Laplace Equation cont.

i. $k=p^2$: $X=c_1e^{px} + c_2e^{-px}$, $Y=c_3\cos py + c_4\sin py$

ii. $k=-p^2$: $X=c_5\cos px + c_6\sin px$, $Y=c_7e^{py} + c_8e^{-py}$

iii. $k=0$: $X=c_9x + c_{10}$, $Y=c_{11}y + c_{12}$

Thus, various possible solutions are:

$u=(c_1e^{px} + c_2e^{-px})(c_3\cos py + c_4\sin py)$

$u=(c_5\cos px + c_6\sin px)(c_7e^{py} + c_8e^{-py})$

$u=(c_9x + c_{10})(c_{11}y + c_{12})$

Laplace Equation cont.

Analytic functions:

- The real and imaginary parts of a complex analytic function both satisfy the Laplace equation.
- If $f(x + iy)=u(x, y) + iv(x, y)$ is an analytic function, then $u_{xx} + u_{yy}=0, v_{xx} + v_{yy}=0$
- The close connection between the Laplace equation and analytic functions implies that any solution of the Laplace equation has derivatives of all orders, and can be expanded in a power series, at least inside a circle that does not enclose a singularity.

Elliptic Partial Differential Equations cont.

(ii) Poisson Equation: $\Delta u + \Phi = 0$

- The two dimensional Poisson equation has the following form:

$u_{xx} + u_{yy} + f(x, y) = 0$ in the Cartesian coordinate system,

$(1/r)(ru_r)_r + (1/r^2)u_{\vartheta\vartheta} + g(r, \vartheta) = 0$ in the polar coordinate system

- Poisson's equation is a partial differential equation with broad utility in electrostatics, mechanical engineering and theoretical physics.
- E.g. In electrostatics: $\Delta V = -\rho/\varepsilon$

Elliptic Partial Differential Equations cont.

(iii) Helmholtz Equation: $\Delta u + \lambda u = -\Phi$

- Many problems related to steady state oscillations (mechanical, acoustical, thermal, electromagnetic) lead to the two dimensional Helmholtz equation. For ¸ $\lambda < 0$, this equation describes mass transfer processes with volume chemical reactions of the first order.

Helmholtz Equation cont.

- The two dimensional Helmholtz equation has the following form:

$u_{xx} + u_{yy} + \lambda u = -f(x, y)$ in the Cartesian coordinate system,

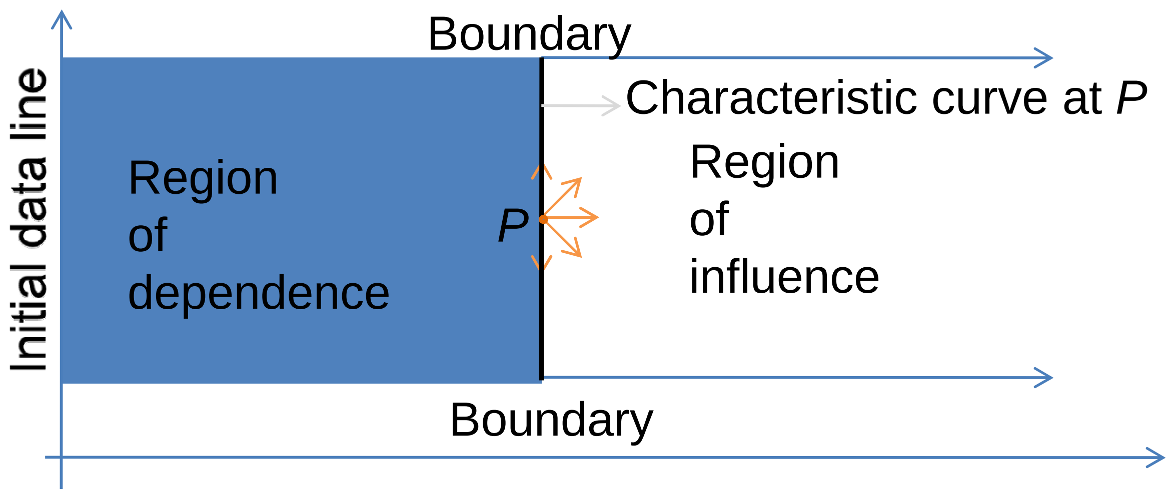
$(1/r)(ru_r)_r + (1/r^2)u_{\vartheta\vartheta} + \lambda u = -g(r, \vartheta)$ in the polar coordinate system

Parabolic Partial Differential Equations

- The discriminant $B^2 - 4AC = 0$
- Equations that are parabolic at every point can be transformed into a form analogous to the heat equation by a change of independent variables.
- Solutions smooth out as the transformed time variable increases

Parabolic Partial Differential Equations cont.

- Region of influence: Part of domain away from initial data line from the characteristic curve
- Region of dependence: Part of domain from the initial data line to the characteristic curve



Parabolic Partial Differential Equations cont.

Examples:

i. $u_t=au_{xx}$ heat equation (linear heat equation)

ii. $u_t=au_{xx} + f(x, t)$ non-homogeneous heat equation

iii. $u_t=au_{xx} + bu_x + cu + f(x, t)$ convective heat equation with a source

iv. $u_t=a(u_{rr} + (1/r)u_r)$ heat equation with axial symmetry

Parabolic Partial Differential Equations cont.

v. $u_t=a(u_{rr} + (1/r)u_r) + g(r, t)$ heat equation with axial symmetry (with a source)

vi. $u_t=a(u_{rr} + (2/r)u_r)$ heat equation with central symmetry

vii. $u_t=a(u_{rr} + (2/r)u_r) + g(r, t)$ heat equation with central symmetry (with a source)

viii. $i\hbar u_t=-(\hbar^2/2m)u_{xx} + h(x)u$ Schrodinger equation (linear schrodinger equation)

Parabolic Partial Differential Equations cont.

- Heat equation: $u_t = a\Delta u$
- The maximum value of u is either earlier in time than the region of concern or on the edge of the region of concern.
- even if u has a discontinuity at an initial time $t = t_0$, the temperature becomes smooth as soon as $t > t_0$. For example, if a bar of metal has temperature 0 and another has temperature 100 and they are stuck together end to end, then very quickly the temperature at the point of connection is 50 and the graph of the temperature is smoothly running from 0 to 100.

Parabolic Partial Differential Equations cont.

Solution of the heat equation:

Consider $u_t=au_{xx}$ (3)

- In plain English, this equation says that the temperature at a given time and point will rise or fall at a rate proportional to the difference between the temperature at that point and the average temperature near that point.

Solve by separation of variables

Let $u(x, t)=X(x)T(t)$

Substituting this in (3), we get

$$X''/X=T'/aT=k$$

Solution of heat equation cont.

i. $k=p^2$: $X=c_1e^{px} + c_2e^{-px}$, $T=c_3e^{ap^2t}$

ii. $k=-p^2$: $X=c_4\cos px + c_5\sin px$, $T=c_6e^{-ap^2t}$

iii. $k=0$: $X=c_7x + c_8$, $T=c_9$

Thus, various possible solutions are:

$u=(c_1e^{px} + c_2e^{-px})(c_3e^{ap^2t})$

$u=(c_4\cos px + c_5\sin px)(c_6e^{-ap^2t})$

$u=(c_7x + c_8)c_9$

Parabolic Partial Differential Equations cont.

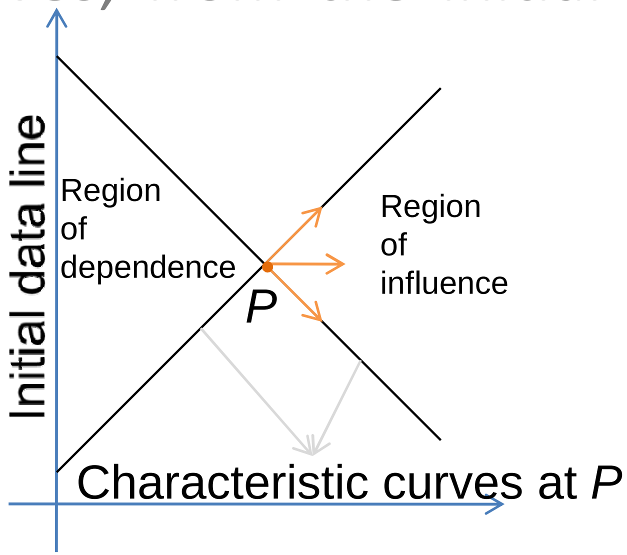
- Let $u(x, t)$ be a continuous function and a solution of $u_t = a u_{xx}$ for $0 \le x \le l$, $0 \le t \le T$, where $T > 0$ is a fixed time. Then the maximum and minimum values of u are attained either at time $t=0$ or at the end points $x=0$ and $x=l$ at some time in the interval $0 \le t \le T$

Hyperbolic Partial Differential Equations

- The discriminant $B^2 - 4AC > 0$
- Hyperbolic equations retain any discontinuities of functions or derivatives in the initial data
- If a disturbance is made in the initial data of a hyperbolic differential equation, then not every point of space feels the disturbance at once. Relative to a fixed time coordinate, disturbances have a finite propagation speed. They travel along the characteristics of the equation
- An example is the wave equation

Hyperbolic Partial Differential Equations cont.

- **Region of influence:** Part of domain, between the characteristic curves, from point *P* to away from the initial data line
- **Region of dependence:** Part of domain, between the characteristic curves, from the initial data line to the point *P*



Hyperbolic Partial Differential Equations cont.

Examples:

i. $u_{tt}=a^2u_{xx}$ wave equation (linear wave equation)

ii. $u_{tt}=a^2u_{xx} + f(x, t)$ non-homogeneous wave equation

iii. $u_{tt}=a^2u_{xx} - bu$ Klein-Gordon equation

iv. $u_{tt}=a^2u_{xx} - bu + f(x, t)$ non-homogeneous Klein-Gordon equation

Hyperbolic Partial Differential Equations cont.

v. $u_{tt}=a^2(u_{rr} + (1/r)u_r) + g(r, t)$ non-homogeneous wave equation with axial symmetry

vi. $u_{tt}=a^2(u_{rr} + (2/r)u_r) + g(r, t)$ non-homogeneous wave equation with central symmetry

vii. $u_{tt} + ku_t=a^2u_{xx} + bw$ Telegraph equation

Hyperbolic Partial Differential Equations cont.

Solution of the wave equation:

Consider $u_{tt}=a^2u_{xx}$ (4)

- The equation has the property that, if u and its first time derivative are arbitrarily specified initial data on the initial line $t = 0$ (with sufficient smoothness properties), then there exists a solution for all time.

D'Alembert's solution

Introduce new independent variables:

$$y=x + at,\ z=x - at$$

Substituting these in (4), we get

$$u_{yz}=0 \quad (5)$$

Solution of Wave Equation cont.

Integrating (5) w.r.t. z, we get $u_y = f(y)$ (6)

Integrating (6) w.r.t. y, we obtain

$$u = \varphi(y) + \psi(z), \text{ where } \varphi(y) = \int f(y)dy$$

Thus, $u(x, t) = \varphi(x + at) + \psi(x - at)$ (7) is the general solution of (4)

Now suppose, $u(x, 0) = g(x)$ and $u_t(x, 0) = 0$, then (7) takes the form $u(x, t) = g(x + at) + g(x - at)$

which is the d'Alembert's solution of the wave equation (4)

Summary

- Second order semi-linear equation in two variables: $A(x, y)u_{xx} + B(x, y)u_{xy} + C(x, y)u_{yy} = \varphi(x, y, u, u_x, u_y)$ classified as

i. Elliptic: $B^2 - 4AC < 0$

ii. Parabolic: $B^2 - 4AC = 0$

iii. Hyperbolic: $B^2 - 4AC > 0$

Summary & Conclusion

General relation between the physical problems and the type of PDEs

- Propagation problems lead to parabolic or hyperbolic PDEs.
- Equilibrium equations lead to elliptic PDE.
- Most fluid equations with an explicit time dependence are Hyperbolic PDEs
- For dissipation problem, Parabolic PDEs

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